ON THE PROBLEM OF STABILITY OF MOTION IN CRITICAL CASES

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An investigation is made of the stability of motion described by the system of equations (0.1)

$$x_s^{\cdot} = -\lambda_s y_s + X_s(x_1, \ldots, x_p; y_1, \ldots, y_p, \tau), \quad y_s^{\cdot} = \lambda_s x_s + Y_s(x_1, \ldots, x_p; y_1, \ldots, y_p, \tau) \, (s = 1, \ldots, p)$$

where X_s and Y_s are holomorphic functions of the variables $x_1, \ldots, x_p$, $y_1, \ldots, y_p$. The expansion of these functions begins with terms not lower than the second order. The coefficients of the expansions of X_s and Y_s are periodic functions with the common real period ω.

In case $p = 1$ and $\lambda\omega/\pi$ is irrational the problem on stability was solved by Liapunov [1].

Below, the problem of Liapunov is solved for rational $\lambda\omega/\pi$. The case of canonical system is investigated and the results are extended to systems of higher order.

1. Investigation of Liapunov's problem for rational $\lambda\omega/\pi$. Let us consider the second order system

$$x^{\cdot} = -\frac{\alpha}{\beta} y + \frac{1}{\beta} \sum_{l=2}^{\infty} X^{(l)}(x, y, \tau), \qquad y^{\cdot} = \frac{\alpha}{\beta} x + \frac{1}{\beta} \sum_{l=2}^{\infty} Y^{(l)}(x, y, \tau) \quad (1.1)$$

where

$$X^{(l)} = \sum_{k_1+k_2=l} a^{(k_1, k_2)}(\tau) x^{k_1} y^{k_2}, \qquad Y^{(l)} = \sum_{k_1+k_2=l} b^{(k_1, k_2)}(\tau) x^{k_1} y^{k_2}$$

$$a^{(k_1, k_2)}(\tau) = \sum_{n=0}^{\infty} (a_{0n}^{(k_1, k_2)} \cos n\tau + a_{1n}^{(k_1, k_2)} \sin n\tau)$$

$$b^{(k_1, k_2)}(\tau) = \sum_{n=0}^{\infty} (b_{0n}^{(k_1, k_2)} \cos n\tau + b_{1n}^{(k_1, k_2)} \sin n\tau)$$

and α and β are positive integers.

Setting $\tau = \beta t$ and passing to the variables ξ and η

$$x = \xi \cos \alpha t + \eta \sin \alpha t, \qquad y = \xi \sin \alpha t - \eta \cos \alpha t$$

we obtain

$$\xi^{\cdot} = \sum_{l=2}^{\infty} P^{(l)}(\xi, \eta, t), \qquad \eta^{\cdot} = \sum_{l=2}^{\infty} Q^{(l)}(\xi, \eta, t) \tag{1.2}$$

$$P^{(l)}(\xi, \eta, t) = X^{(l)} \cos\alpha t + Y^{(l)} \sin\alpha t = \sum_{k_1+k_2=l} A^{(k_1, k_2)}(t)\xi^{k_1}\eta^{k_2}$$

$$Q^{(l)}(\xi, \eta, t) = X^{(l)} \sin\alpha t - Y^{(l)} \cos\alpha t = \sum_{k_1+k_2=l} B^{(k_1 k_2)}(t)\xi^{k_1}\eta^{k_2}$$

$$A(t) = A(t+2\pi) \qquad B(t) = B(t+2\pi)$$

Transforming Equation (1.2) and setting

$$x_1 = \xi + \sum_{k_1+k_2=2}^{N_1} u^{(k_1, k_2)}(t)\xi^{k_1}\eta^{k_2}, \qquad y_1 = \eta + \sum_{k_1+k_2=2}^{N_1} v^{(k_1, k_2)}(t)\xi^{k_1}\eta^{k_2}$$

where $u^{(k_1,k_2)}(t)$ and $v^{(k_1,k_2)}(t)$ are periodic functions of t which have to be defined, in consequence we obtain

$$x_1^{\cdot} = \sum_{l=2}^{\infty} X_1^{(l)}(x_1, y_1, t), \qquad y_1^{\cdot} = \sum_{l=2}^{\infty} Y_1^{(l)}(x_1, y_1, t) \tag{1.3}$$

where

$$X_1^{(l)} = \sum_{k_1+k_2=l} \alpha^{(k_1, k_2)} x_1^{k_1} y_1^{k_2}, \qquad Y_1^{(l)} = \sum_{k_1+k_2=l} \beta^{(k_1, k_2)} x_1^{k_1} y_1^{k_2} \tag{1.4}$$

$$\alpha^{(k_1, k_2)} = A^{(k_1, k_2)} + \varphi^{(k_1, k_2)} + \frac{du^{(k_1, k_2)}}{dt}, \qquad \beta^{(k_1, k_2)} = B^{(k_1,k_2)} + \psi^{(k_1, k_2)} + \frac{dv^{(k_1, k_2)}}{dt}$$

For values $k_1 + k_2 = 2$, the functions $\varphi^{(k_1,k_2)}$ and $\psi^{(k_1,k_2)}$ are equal to zero, and for the values $k_1 + k_2 > 2$, they are known functions $u^{(k_1, k_2)}$, $v^{(k_1,k_2)}$, $A^{(k_1,k_2)}$, $B^{(k_1,k_2)}$, in which $k_1 + k_2 < l$.

Let us define the periodic functions $u^{(k_1,k_2)}$ and $v^{(k_1,k_2)}$ so that the quantities $\alpha^{(k_1,k_2)}$ and $\beta^{(k_1,k_2)}$ should be constant. For this it is necessary and sufficient to define these quantities by means of Equations (1.5)

$$\alpha^{(k_1, k_2)} = \frac{1}{2\pi}\int_0^{2\pi} [A^{(k_1, k_2)} + \varphi^{(k_1, k_2)}]\, dt, \quad \beta^{(k_1, k_2)} = \frac{1}{2\pi}\int_0^{2\pi} [B^{(k_1, k_2)} + \psi^{(k_1, k_2)}]\, dt$$

Then the functions $u^{(k_1,k_2)}$ and $v^{(k_1,k_2)}$ will be defined as follows:

$$u^{(k_1, k_2)} = \int [\alpha^{(k_1, k_2)} - A^{(k_1, k_2)} - \varphi^{(k_1, k_2)}]\, dt$$

$$v^{(k_1, k_2)} = \int [\beta^{(k_1, k_2)} - B^{(k_1, k_2)} - \psi^{(k_1, k_2)}]\, dt$$

We thus have the result

$$\begin{aligned} x_1^{\cdot} &= X_1^{(m)}(x_1, y_1) + X_1^{(m+1)}(x_1, y_1) + \ldots + X_1^{(m+N)}(x_1, y_1) + \\ &\quad + X_1^{(m+N+1)}(x_1, y_1, t) \\ y_1^{\cdot} &= Y_1^{(m)}(x_1, y_1) + Y_1^{(m+1)}(x_1, y_1) + \ldots + Y_1^{(m+N)}(x_1, y_1) + \\ &\quad + Y_1^{(m+N+1)}(x_1, y_1, t) \end{aligned} \tag{1.6}$$

Here $m \geqslant 2$, $m + N = N_1$ and is equal to some arbitrary large number. It is obvious that the problems on the stability with respect to the variables x_1, y_1 and x and y are equivalent. In what follows the subscript 1 in the system (1.6) will be omitted.

Hence, Liapunov's problem for rational $\lambda\omega/\pi$ can be reduced to the problem on the stability of the self-contained system with two zero roots if the problem on the stability can be solved by means of a finite number of terms on the right-hand sides of the system (1.6). This system was considered in the papers [2] and [3].

In the above papers, there were formulated necessary and sufficient conditions for the stability of the integrals of the system (1.6) in the simplest case when the problem is solved by forms of the mth order.

Let us formulate these conditions. If the forms $X^{(m)}$ and $Y^{(m)}$ are such that Equation

$$F_0 = xY^{(m)} - yX^{(m)} = 0$$

has real solutions

$$a_k x + b_k y = 0 \qquad (k = 1, \ldots, p) \qquad\qquad p \leqslant m + 1$$

and if at least on one straight line $a_k x + b_k y = 0$ the form

$$R_0 = xX^{(m)} + yY^{(m)}$$

can take on positive values, then the unperturbed motion is unstable. If, however, $R_0 < 0$ on all straight lines then the motion is asymptotically stable.

If Equation $F_0 = 0$ has no real solutions different from $x = y = 0$, the problem on the stability is solved by the sign of the expression

$$g = F_0(\cos\theta, \sin\theta) \int_0^{2\pi} \frac{R_0(\cos\theta, \sin\theta)}{F_0(\cos\theta, \sin\theta)}\, d\theta$$

If $g < 0$, the unperturbed motion is asymptotically stable; if, however, $g > 0$, it is unstable.

The case $g = 0$, and the case when for $F_0 = 0$ the form R_0 is negative or may vanish on one or several straight lines, are indeterminate. The solution represents no difficulties when $g = 0$ and it has been completely investigated in papers [2 and 3] .

Returning now to the system (1.6), and applying to it the formulated criteria of stability, we see that Liapunov's problem is solved in two cases:

1) when the problem of stability is solved by forms of the mth order independently of higher order forms,

2) when the form $F_0 = xY^{(m)} - yX^{(m)}$ is of a definite sign.

However, if the form F_0 is not of a definite sign and if the form $R_0 \leqslant 0$ when $F_0 = 0$, then the problem on the stability depends on a study of forms of higher order than the mth order.

2. Stability criteria involving forms of the (m + 1) order. The vanishing of the forms F_0 and R_0 on the straight lines $a_k x + b_k y = 0$ can occur only when the forms $X^{(m)}$ and $Y^{(m)}$ have the common factor $a_k x + b_k y$. Hence, in the indeterminate case $F_0 = 0$ and $R \leqslant 0$, these forms can be expressed in the form

$$X^{(m)} = \prod_{j=1}^{p} (a_j x + b_j y)^{\nu_j} X^{(m-k)}$$

$$Y^{(m)} = \prod_{j=1}^{p} (a_j x + b_j y)^{\nu_j} Y^{(m-k)} (\nu_1 + \ldots + \nu_p) = k \tag{2.1}$$

Here, $X^{(m-k)}$ and $Y^{(m-k)}$ are forms of order $(m - k)$ and they do not have common factors of the type $ax + by$.

The functions F_0 and R_0 will have the from

$$F_0 = \prod_{j=1}^{p} (a_j x + b_j y)^{\nu_j} F_{-k}(x, y), \qquad R_0 = \prod_{j=1}^{p} (a_j x + b_j y)^{\nu_j} R_{-k}(x, y) \tag{2.2}$$

$$F_{-k} = xY^{(m-k)} - yX^{(m-k)}, \qquad R_{-k} = xX^{(m-k)} + yY^{(m-k)}$$

If the common real roots of Equations $X^{(m)}(1, \varkappa) = 0$ and $Y^{(m)}(1, \varkappa) = 0$ are denoted by $\varkappa_1, \ldots, \varkappa_p$, then the common factors $a_j x + b_j y$ will have the form $y + \varkappa_j x$, $(\varkappa_j = -a_j / b_j)$.

Let us first study the system (1.6) assuming that in (2.1)

$$\nu_1 = \nu_2 = \ldots = \nu_p = 1, \qquad F_{-k}(\varkappa_j) \neq 0 \qquad (j = 1, \ldots, p)$$

We construct for each straight line $-y + \varkappa_j x = 0$ the function

$$\Phi_j = \frac{Y^{(m-k)}(x, y)\, X^{(m+1)}(x, y) - X^{(m-k)}(x, y)\, Y^{(m+1)}(x, y)}{xY^{(m-k)} - yX^{(m-k)}} \tag{2.3}$$

The conditions of stability can be formulated in terms of order $(m + 1)$ in the following way.

If on at least one of the straight lines $y + \varkappa_j x = 0$ $(j = 1, \ldots, p)$ the function Φ_j takes on a positive value, then the unperturbed motion is unstable.

If, however, $\Phi_j < 0$ on all straight lines then the unperturbed motion is asymptotically stable.

Let us assume that $\Phi_1 > 0$. Expressing the system (1.6) in the form

$$x^{\bullet} = (-y + \varkappa_1 x) X^{(m-1)} + X^{(m+1)} + \ldots$$

$$y^{\bullet} = (-y + \varkappa_1 x) Y^{(m-1)} + Y^{(m+1)} + \ldots$$

and setting $y_1 = -y + \varkappa_1 x$, we have

$$x^{\bullet} = y_1 X_{*}^{(m-1)}(x, y_1) + X_{*}^{(m+1)}(x, y_1) + \ldots$$

$$y^{\bullet}_1 = y_1 Y_{*}^{(m-1)}(x, y_1) + Y_{*}^{(m+1)}(x, y_1) + \ldots \tag{2.4}$$

$$X_*^{(m+l)} = \sum_{k=0}^{m+l} A_{*k}^{(m+l)} x^{m+l-k} y_1{}^k, \qquad Y_*^{(m+l)} = \sum_{k=0}^{m+l} B_{*k}^{(m+l)} x^{m+l-k} y_1{}^k \quad \text{(2.4) cont.}$$

$$A_{*k}^{(m+l)} = \frac{(-1)^k}{k!} \frac{d^k X^{(m+l)}(1, \varkappa_1)}{d\varkappa_1{}^k}$$

$$B_{*k}^{(m+l)} = \frac{(-1)^{k+1}}{k!} \left[\frac{d^k Y^{(m+l)}(1, \varkappa_1)}{d\varkappa_1{}^k} - \varkappa_1 \frac{d^k X^{(m+l)}(1, \varkappa_1)}{d\varkappa_1{}^k} \right] \qquad (l = -1, 1, 2, \ldots)$$

We note that

$$B_{*0}^{(m-1)} = [Y^{(m-1)}(1, \varkappa_1) - \varkappa_1 X^{(m-1)}(1, \varkappa_1)] \neq 0$$

Let us transform the system (2.4) by setting

$$x_1 = x - \mu y_1, \quad \mu = A_{*0}^{(m-1)} / B_{*0}^{(m-1)}$$

As a result we obtain

$$x_1^{\cdot} = y_1 X_1^{(m-1)}(x_1, y_1) + X_1^{(m+1)}(x_1, y_1) + \ldots$$

$$y_1^{\cdot} = y_1 Y_1^{(m-1)}(x_1, y_1) + Y_1^{(m+1)}(x_1, y_1) + \ldots \qquad (2.5)$$

$$X_1^{(m+l)} = \sum A_k^{(m+l)} x_1^{m+l-k} y_1^k, \quad Y_1^{(m+l)} = \sum B_k^{(m+l)} x_1^{m+l-k} y_1^k$$

$$A_{m+l-q}^{(m+l)} = \frac{1}{q!} \left[\frac{d^q X_*^{(m+l)}(\mu, 1)}{d\mu^q} - \mu \frac{d^q Y_*^{(m+l)}(\mu, 1)}{d\mu^q} \right], \quad B_{m+l-q}^{(m+l)} = \frac{1}{q!} \frac{d^q Y_*^{(m+l)}(\mu, 1)}{d\mu^q}$$

We note that

$$A_0^{(m+1)} = \Phi_1(1, \varkappa_1), \qquad B_0^{(m-1)} = -F_{-1}(1, \varkappa_1), \qquad A_0{}^{(m-1)} = 0$$

Without restricting the generality of the problem, one may set $B_0^{(m+l)} = 0$ for $l = 1, \ldots, N$. If these coefficients are not zero, one can make them zero for $l \leqslant N$, by a change of variables

$$y_1 = z + a_2 x_1^2 + \ldots + a_N x_1^N$$

and the proper choice of the numbers $a_2, \ldots, a_N$. During this transformation the coefficients of the forms $X_1{}^{(m-1)}$ and $X_1^{(m+1)}$ will not change. The series

$$y_1(x_1) = c_2 x_1^2 + c_3 x_1^3 + \ldots \qquad (2.6)$$

will satisfy formally the equation derived from the system (2.5) by eliminating the time t . These series are usually divergent.

As a consequence of the transformation, the system (1.6) takes on the form

$$x_1^{\cdot} = z^2 (A_1{}^{(m-1)} x_1{}^{m-2} + \ldots + A_{m-1}^{(m-1)} z^{m-2}) + A_0{}^{(m+1)} x_1{}^{m+1} + \ldots$$
$$\ldots + A_{m+1}^{(m+1)} z^{m+1} + \sum_{l=2}^{\infty} \sum_{k=0}^{m+l} H_k{}^{(m+l)} x_1{}^{m+l-k} z^k \qquad (2.7)$$
$$z^{\cdot} = z (B_0{}^{(m-1)} x_1{}^{m-1} + \ldots + B_{m-1}^{(m-1)} z^{m-1}) + \sum_{l=1}^{\infty} \sum_{k=0}^{m+l} E_k{}^{(m+l)} x_1{}^{m+l-k} z^k$$

For this system one can construct functions V and W which satisfy the theorem of Chetaev [4]. Let us set $V = x_1^2 + z^2$. In the region $x_1^4 - z^2 > 0$, $x_1 > 0$, we shall have $V' > 0$.

For the function W in Chetaev's theorem, we choose the function $W = x_1^{2k} - z^2$, $k > 2$. The region $W > 0$ is contained inside the region $VV' > 0$. It is easy to see that the sign of W' on the boundary of the region $W > 0$ on which $W = 0$ is determined by the sign of the expression $-2B_0^{(m-1)} z^2 x_1^{(m-1)}$, whose sign is invariable if $x_1 > 0$. Hence, the unperturbed motion described by the system (1.6) is unstable when $\Phi_1 > 0$.

Above we have considered the straight line $-y + \varkappa_1 x = 0$. We can concider any other straight line $-y + \varkappa_j x = 0$ in an analogous way.

Let us prove now that if $\Phi_j(x, y) < 0$ $(j = 1, \ldots, p)$, on every line $a_j x + b_j y = 0$ then the unperturbed motion is asymptotically stable.

Transforming the system (1.6) to polar coordinates, we obtain

$$r^{\cdot} = r^m R_{-k} \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta) + r^{m+1} R_1 + \ldots,$$

$$\theta^{\cdot} = r^{m-1} F_{-k} \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta) + r^m F_1 + \ldots \tag{2.8}$$

$$R_l = X^{(m+l)}(\cos\theta, \sin\theta)\cos\theta + Y^{(m+l)}(\cos\theta, \sin\theta)\sin\theta$$
$$F_l = Y^{(m+l)}(\cos\theta, \sin\theta)\cos\theta - X^{(m+l)}(\cos\theta, \sin\theta)\sin\theta \qquad (l = -k, 1, 2, \ldots)$$

Let us first consider the case when $F_{-k}(1, \varkappa)$ has no real roots. Liapunov's function which corresponds to the system (2.8) may be taken in the form

$$V = r \exp \int_0^\theta \psi(\theta)\, d\theta \tag{2.9}$$

if one defines the function $\psi(\theta)$ by means of Equation

$$R_{-k} + \psi F_{-k} = -h(\theta) \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta) \tag{2.10}$$

where $h(\theta)$ is a bounded, continuous, positive and periodic function of θ with period 2π; obviously, $h(\theta)$ cannot vanish for any real value of θ

In order to insure the periodicity of $\psi(\theta)$ we must impose the following condition on $h(\theta)$

$$\int_0^{2\pi} \frac{1}{F_{-k}(\theta)} h(\theta) \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta)\, d\theta = -\int_0^{2\pi} \frac{R_{-k}(\theta)}{F_{-k}(\theta)}\, d\theta$$

The derivative V' will have the form

$$V' = r^m \exp \int_0^\theta \psi(\theta)\, d\theta \Big\{ -h(\theta) \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta)^2 +$$
$$+ \frac{r}{F_{-k}} \Big[F_{-k} R_1 - R_{-k} F_1 - F_1 h \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta) \Big] + \ldots \Big\}$$

For values of θ different from the values θ_j, which are determined by Equations $\varkappa_j \cos\theta - \sin\theta = 0$, the derivative V' takes only negative values. For the values $\theta = \theta_s$, the sign of the second term, contained in the square brackets, coincides with the sign of the expression

$$\frac{F_{-k}R_1 - R_{-k}F_1}{F_{-k}} = \frac{X^{(m+1)}Y^{(m-k)} - Y^{(m+1)}X^{(m-k)}}{\cos\theta\, Y^{(m-k)} - \sin\theta\, X^{(m-k)}}$$

which is negative by hypothesis. Therefore, V' is a negative definite function of r for arbitrary values of θ. This implies the asymptotic stability of the unperturbed motion.

Let us suppose that Equation $F_{-k}(1, \varkappa) = 0$ has real roots different from the common real roots of $X^{(m)}(1, \varkappa) = 0$ and $Y^{(m)}(1, \varkappa) = 0$. Let these roots have the values μ_s $(s = 1, \ldots, q)$.

Then Expression

$$R_0 = R_{-k} \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta)$$

will be negative for values of θ determined by Equations $\mu_s \cos\theta - \sin\theta = 0$. In the opposite case we will have unstable integral curves.

Let us now determine the function $\psi(\theta)$ in the interval $(0, 2\pi)$, and, hence for all real values of θ, by means of Equation (2.10). We do this in the following way: we set $\psi \equiv 0$ in the intervals

$$\theta_{\mu_s} - \varepsilon \leqslant \theta \leqslant \theta_{\mu_s} + \varepsilon.$$

For this it is sufficient to define $h(\theta)$ by means of Equation

$$R_{-k} = -h(\theta) \prod_{j=1}^{p} (\varkappa_j \cos\theta - \sin\theta)$$

Such a definition $h(\theta)$ is possible because $R_0 < 0$ in the intervals $\theta_{\mu_s} - \varepsilon \leqslant \theta \leqslant \theta_{\mu_s} + \varepsilon$. On the function $h(\theta)$ in the interval $(0, 2\pi)$ we impose the condition

$$\int_0^{2\pi} \psi(\theta)\, d\theta = 0$$

which can always be satisfied if $h(0) = h(2\pi)$.

For such a choice of the function ψ, the derivative V' will be negative-definite for real θ. Therefore, the unperturbed motion is asymptotically stable.

Let us consider the case when $\nu_j \geqslant 1$ $(j = 1, \ldots, p)$. This can occur when Equations $X^{(m)}(1, \varkappa) = 0$ and $Y^{(m)}(1, \varkappa) = 0$ have common multiple real roots. We shall give the conditions for stability in terms of forms of the order $m + 1$.

The unperturbed motion is asymptotically stable for arbitrary $\nu_1, \ldots, \nu_p$ if the forms (2.1) are such that:

1) the Equation $F_{-k}(1, \varkappa) = 0$ has no roots equal to $\varkappa_1, \ldots, \varkappa_p$ but has at least one real root $\varkappa^0$;

2) the functions $\Phi_j < 0$ on all straight lines $y = \varkappa_j x$ $(j = 1, \ldots, p)$.

The unperturbed motion is also asymptotically stable if the equation $F_{-k}(1, \varkappa) = 0$ has no real solution but there is among the numbers $\nu_1, \ldots, \nu_p$ at least one odd number, and if hereby $\Phi_j < 0$ on all lines $y = \varkappa_j x$.

If $\nu_1, \ldots, \nu_p$ are even numbers, and if Equation $F_{-k}(1, \varkappa) = 0$ has no real solutions, then the unperturbed motion will be asymptotically stable when the inequality

$$F(\cos\theta, \sin\theta)\int_0^{2\pi} \frac{R_{-k}(\cos\theta, \sin\theta)}{F_{-k}(\cos\theta, \sin\theta)}\, d\theta < 0 \qquad \text{for } \Phi_j < 0,\ y = \varkappa_j x$$

is valid.

We note that in the case of even $\nu_1, \ldots, \nu_p$ one may assume that the form $R_{-k}(x, y) < 0$ for $y = \varkappa^\circ x$.

The proof of these assertions is analogous to the one presented above, except for the case when $\nu_1, \ldots, \nu_p$ are even and Equation $F_{-k}(1, \varkappa) = 0$ has no real roots. Let us investigate this case. We take the Liapunov function as before. The function $\psi(\theta)$ is determined by Equations

$$R_{-k} + \psi F_{-k} = -h(\theta)\prod_{j=1}^{p}(\varkappa_j \cos\theta - \sin\theta)^2$$

The condition of periodicity of the function $\psi(\theta)$ takes the form

$$\int_0^{2\pi} \frac{1}{F_{-k}}\, h(\theta)\prod_{j=1}^{p}(\varkappa_j\cos\theta - \sin\theta)^2\, d\theta = -\int_0^{2\pi}\frac{R_{-k}(\theta)}{F_{-k}(\theta)}\, d\theta$$

This condition is always satisfied with the corresponding choice of $h(\theta) > 0$ if

$$F_{-k}\int_0^{2\pi}\frac{R_{-k}}{F_{-k}}\, d\theta < 0$$

The rest of the proof is analogous to the case when $\nu_1 = \nu_2 = \ldots = \nu_p = 1$.

Next, we consider the case $\Phi_j > 0$, $\nu_j \geqslant 2$. In the solving of this problem we limit ourselves to the case of one multiple root $\varkappa_1$ $(p = 1)$.

In this case the system (2.8) will have the form

$$r^{\cdot} = r^m(\varkappa_1\cos\theta - \sin\theta)^{\nu_1}R_{-\nu_1} + r^{m+1}R_1 + \ldots$$
$$\theta^{\cdot} = r^{m-1}(\varkappa_1\cos\theta - \sin\theta)^{\nu_1}F_{-\nu_1} + r^m F_1 + \ldots$$

Let us suppose that the mth order forms determine the nonasymptotically stable integral curves and let us take the Liapunov function in its earlier form (2.9). Let ν_1 be an odd number. We determine the $\psi(\theta)$ by means of Equation

$$R_{-\nu_1} + \psi F_{-\nu_1} = h(\theta)(\varkappa_1\cos\theta - \sin\theta)$$

The function $h(\theta) > 0$ is found from the condition of periodicity of the function $\psi(\theta)$. Then when $\Phi_1 > 0$, the derivative V' will be positive definite which guarantees the instability of the unperturbed motion.

If ν_1 is an even number and if Equation $F_{-\nu_1}(1, \varkappa) = 0$ has no real roots, then when $\Phi_1^{\cdot} > 0$ the unperturbed motion will be unstable if

$$F_{-\nu_1}(\cos\theta, \sin\theta)\int_0^{2\pi}\frac{R_{-\nu_1}(\cos\theta, \sin\theta)}{F_{-\nu_1}(\cos\theta, \sin\theta)}\,d\theta > 0$$

The proof of this proposition is analogous to the one given above.

Following the arguments presented in Section 2, one can study the case $\nu_j \geqslant 2$ in greater detail. Because of the particular nature of such systems we shall not dwell on this any longer.

3. Stability criteria in terms of higher order forms. Let us consider the unsettled case when $\Phi_j = 0$ on the straight lines $-y + \varkappa_j x = 0$ $(j = 1, \ldots, p_1)$ and takes on negative values on such lines with $(j = p_1 + 1, \ldots, p)$ $(\nu_1 = \nu_2 = \ldots = \nu_p = 1)$.

It is quite complicated to formulate the criteria of stability on the basis of the structure of the right-hand sides of Equations (1.6). Therefore, in what follows, we shall give the formulations in reference to the system (2.7).

We note that for any transformation $y_j = -y + \varkappa_j x$ we shall obtain systems analogous to (2.7). Let us assume that in one of these systems $\Phi_j = 0$, while the first nonvanishing coefficient $H_0^{(m+l)}$ has an index $l = \alpha_j < N$. We will prove that in case $H_0^{(m+\alpha_j)} > 0$ the unperturbed motion is unstable. Let us take the Chetaev function in the form $V = x_1^2 + z^2$. The sign of its derivative in the region $-z^2 + x_1^{2k} > 0$, $x_1 > 0$ when $2k = 3 + \alpha_j$ is determined by the sign of Expression $H_0^{(m+\alpha_j)} x_1^{m+\alpha_j+1}$, which is positive when $x_1 > 0$. Hence, VV' will be positive in the selected region.

In the region where $VV' > 0$ we take the function $W = -z^2 + x_1^{2(k+1)}$. The derivative of this function preserves an invariable sign when $W = 0$. This establishes the instability of the unperturbed motion.

If we find that the coefficients $H_0^{(m+\alpha_j)} < 0$, on the lines $-y + \varkappa_j x = 0$ $(j = 1, \ldots, p)$ on which $\Phi_j = 0$, then the unperturbed motion is asymptotically stable. The proof of this proposition is basically the same as the proof presented in Section 2, therefore we shall not give it here.

It may happen that, no matter how large N may be, the nonvanishing coefficients $H_0^{(m+l)}$ have superscripts with $l > N$. This case is essentially a singular case, and if it is possible to establish this for some one of the lines $-y + \varkappa_j x = 0$, then the series (2.6) will be convergent and it will represent a root of Equation

$$y_1 Y^{(m-1)}(x_1, y_1) + Y^{(m+1)}(x_1, y_1) + \ldots = 0$$

With the substitution $y_1 = z + c_2 x_1^2 + c_3 x_1^3 + \ldots$ the system (2.5) can be reduced to the form

$$x_1^{\cdot} = z[X_*^{(m-1)}(x_1, z) + X_*^{(m+1)}(x_1, z) + X_*^{(m+2)}(x_1, z) + \ldots]$$
$$z^{\cdot} = z[Y_*^{(m-1)}(x_1, z) + Y_*^{(m+1)}(x_1, z) + Y_*^{(m+2)}(x_1, z) + \ldots]$$

This system can be investigated in an analogous way taking into account that $x = 0$ is a singular line.

We note that if the form F_{-1} is of a definite sign, the integrals of this system are always stable, but they are not asymptotically stable.

4. Investigation of the case $X^{(m)}(1,\varkappa_1) = 0$, $Y^{(m)}(1,\varkappa_1) = 0$, $F_{-k}(1,\varkappa_1) = 0$. Let us consider the case when the common real root of Equations $X^{(m)}(1,\varkappa_1) = 0$, $Y^{(m)}(1,\varkappa_1) = 0$ is also a root of Equation $F_{-1}(1,\varkappa_1) = 0$. This case, because of its particular nature, will not be investigated in all of its details here. We shall restrict ourselves to the assumption that the common real root of

$$X^{(m)}(1,\varkappa_1) = 0, \qquad Y^{(m)}(1,\varkappa_1) = 0, \qquad F_{-k}(1,\varkappa_1) = 0$$

is not a root of Equations

$$Y^{(m+1)}(1,\varkappa) - \varkappa X^{(m+1)}(1,\varkappa) = 0,$$

$$3X^{(m-1)}(1,\varkappa) - 2\left[\frac{dY^{(m-1)}(1,\varkappa)}{d\varkappa} - \varkappa\frac{dX^{(m-1)}(1,\varkappa)}{d\varkappa}\right] = 0$$

We shall prove that the integrals of the system of equations (1.6) are always unstable under these assumptions.

Setting $y_1 = -y + \varkappa_1 x$ in the system, we obtain the system (2.4) in which $B_{*0}^{(m-1)} = 0$, $A_{*0}^{(m-1)} \neq 0$. Equation $A_{*0}^{(m-1)} = 0$ may hold in case $\nu_1 \geqslant 2$. Eliminating dt in the system (2.4), we obtain Equation

$$\frac{dy_1}{dx} = \frac{y_1^2(B_{*1}^{(m-1)}x^{m-2} + \ldots + B_{*m-1}^{(m-1)}y_1^{m-2}) + B_{*0}^{(m+1)}x^{m+1} + \ldots}{y_1(A_{*0}^{(m-1)}x^{m-1} + \ldots + A_{*m-1}^{(m-1)}y_1^{m-1}) + A_{*0}^{m+1}x^{m+1} + \ldots} \tag{4.1}$$

By means of the substitution $y_1 = [z(x) + h]\,x^{3/2}$ this equation can be reduced to the form

$$x\frac{dz}{dx} = \delta z + x\varphi_1(x, z) + x^{1/2}\varphi_2(x, z) \tag{4.2}$$

if the number h is determined by means of Equation

$$[{}^3/_2 A_{*0}^{(m-1)} - B_{*1}^{(m-1)}]\,h^2 = B_{*0}^{(m+1)}$$

A real solution for h can be obtained when the sign of the difference standing in the square brackets coincides with the sign of $B_{*0}^{(m+1)}$. This can be accomplished for any system by repalcing x by $-x$ in case m is even, and by repalcing x by $-x$ and y by $-y$ if m is odd.

It is easy to show that Equation (4.2), with δ not equal to a positive integer, has the holomorphic integral

$$z(x) = \sum_{k=1}^{\infty} C_k x^k + x^{1/2}\sum_{k=1}^{\infty} D_k x^k \tag{4.3}$$

Substituting the value $y_1 = [z(x) + h]\,x^{3/2}$ in the first equation of the system (2.4) we shall have

$$\frac{dx}{dt} = A_{*0}^{(m-1)}hx^{m+1/2} + x^{m+1}f(x, x^{1/2}) \tag{4.4}$$

If we select the number h so that $A_{*0}^{(m-1)}h$ is positive, then it follows from (4.4) that the unperturbed motion is unstable. In case δ is a positive integer the integral (4.3) is holomorphic with respect to x, $x^{\frac{1}{2}}$ and $x \ln x$. The derivations are the same as before.

In conclusion we shall consider the case $F_0 \equiv 0$. This identity can exist only when $X^{(m)} = xX^{(m-1)}(x, y)$, $Y^{(m)} = yX^{(m-1)}(x, y)$. If the form $X^{(m-1)}(x, y)$ can take on positive values then the unperturbed motion is unstable since $R = (x^2 + y^2) X^{(m-1)}(x, y)$.

Therefore, stability can exist when the form $X^{(m-1)}(x, y)$ is negative-definite, or if it is represented as

$$X^{(m-1)}(x, y) = -\prod_{j=1}^{p} (a_j x + b_j y)^{2\nu_j}$$

This case has been treated by us already.

If $X^{(m-1)}(x, y)$ is negative-definite, then the unperturbed motion will be asymptotically stable independently of any forms of higher order. For this type of systems, Liapunov's function can be taken in the form $V = x^2 + y^2$.

The contents of Sections 2 to 4 solve Liapunov's problem in more general cases.

5. Canonical systems. As an application, let us investigate the oscillations of a Hamiltonian system with the function

$$H = \frac{\alpha}{23}(x^2 + y^2) + \frac{1}{\beta}\,[(a^{(3,0)}x^3 + a^{(2,1)}x^2 y + a^{(1,2)}xy^2 + a^{(0,3)}y^3 + + H^{(4)}(x, y, \tau) + \ldots] \tag{5.1}$$

where

$$a^{(k_1, k_2)} = \sum (\delta_n^{(k_1, k_2)} \cos n\tau + \gamma_n^{(k_1, k_2)} \sin n\tau), \qquad H^{(l)} = \sum_{k_1 + k_2 = l} a^{(k_1, k_2)} x^{k_1} y^{k_2}$$

$\delta_n^{(k_1, k_2)}$, $\gamma_n^{(k_1, k_2)}$ are real constants.

Special cases of this problem were treated in the papers of Levi-Civita [5], Siegel [6] and Merman [7].

A system of equations with Hamilton's function of the type (5.1) can be represented in the form

$$\begin{aligned} x^{\cdot} &= -\alpha y - a^{(2,1)}x^2 - 2a^{(1,2)}xy - 3a^{(0,3)}y^2 - \frac{\partial H^{(4)}(x, y, t)}{\partial y} - \ldots \\ y^{\cdot} &= \alpha x + 3a^{(3,0)}x^2 + 2a^{(2,1)}xy + a^{(1,2)}y^2 + \frac{\partial H^{(4)}(x, y, t)}{\partial x} + \ldots \end{aligned} \qquad (\tau = \beta t) \quad (5.2)$$

Transforming this system in accordance with Section 1, we obtain (5.3)

$$\begin{aligned} x_1^{\cdot} &= \alpha^{(2,0)} x_1^2 + \alpha^{(1,1)} x_1 y_1 + \alpha^{(0,2)} y_1^2 + X_1^3(x_1, y_1) + \ldots + X_1^{(N)}(x_1, y_1) + \\ &\quad + X_1^{(N+1)}(x_1, y_1, t) + \ldots, \\ y_1^{\cdot} &= \beta^{(2,0)} x_1^2 + \beta^{(1,1)} x_1 y_1 + \beta^{(0,2)} y_1^2 + Y_1^3(x_1, y_1) + \ldots + Y_1^{(N)}(x_1, y_1) + \\ &\quad + Y_1^{(N+1)}(x_1, y_1, t) + \ldots \end{aligned}$$

The constants α^{k_1, k_2} and β^{k_1, k_2} $(k_1 + k_2 = 2)$ will satisfy the conditions

$$\beta^{(1,1)} = -2\alpha^{(2,0)}, \quad \alpha^{(1,1)} = -2\beta^{(0,2)} \tag{5.4}$$
$$\alpha^{(2,0)} = {}^1/_8\,(-3\gamma_1^{(3,0)} + 3\gamma_3^{(3,0)} - 3\gamma_3^{(1,2)} + \gamma_1^{(1,2)} - \delta_1^{(2,1)} - 3\delta_3^{(2,1)} - 3\delta_1^{(0,3)} + 3\delta_3^{(0,3)})$$
$$\beta^{(0,2)} = {}^1/_8\,(-3\delta_1^{(3,0)} + 3\delta_3^{(3,0)} - \delta_1^{(1,2)} - 3\delta_3^{(1,2)} - \gamma_1^{(2,1)} + 3\gamma_3^{(2,1)} - 3\gamma_1^{(0,3)} - 3\gamma_3^{(0,5)})$$
$$\alpha^{(0,2)} = {}^3/_8\,(3\gamma_1^{(3,0)} - \gamma_3^{(3,0)} + \gamma_1^{(1,2)} + \gamma_3^{(1,2)} - \delta_1^{(2,1)} + \delta_3^{(2,1)} - 3\delta_1^{(0,3)} - \delta_3^{(0,3)})$$
$$\beta^{(2,0)} = {}^3/_8\,(-\delta_3^{(3,0)} - 3\delta_1^{(3,0)} - \delta_1^{(1,2)} + \delta_3^{(1,2)} - \gamma_1^{(2,1)} - \gamma_3^{(2,1)} - 3\gamma_1^{(0,3)} + \gamma_3^{(0,3)})$$

in view of the canonical systems (5.2).

In accordance with the criteria of stability with respect to forms of the mth order, the integrals of the system (5.3) for $X^{(l)} = Y^{(l)} \equiv 0$ $(l \geqslant 3)$ can be stable, in general, if

1) $\alpha^{(k_1, k_2)} = \beta^{(k_1, k_2)} = 0 \qquad (k_1 + k_2 = 2)$
2) the forms of the second order have a common factor of the form $ax + by$, and the form $F_1(x, y)$ is of a definite sign.

In all other cases the motion is unstable. It is easy to prove that for the canonical systems the second case can not occur because of conditions $\beta^{(1,1)} = -2\alpha^{(2,0)}$ and $\alpha^{(1,1)} = -2\beta^{(0,2)}$,

The first case can arise when $\beta > 3\alpha$ or when the conditions (5.4) lead to $\alpha^{(k_1, k_2)} = \beta^{(k_1, k_2)} = 0$ for nonvanishing $\delta_1^{(k_1, k_2)}$, $\gamma_1^{(k_1, k_2)}$, $\delta_3^{(k_1, k_2)}$ and $\gamma_3^{(k_1, k_2)}$ In this case the problem of stability can be solved by means of forms of higher order. Let us assume that the lowest forms, that are not identically zero, are $X^{(m)}$ and $Y^{(m)}$. Then stability with respect to mth order forms can occur only in two cases:

1) the form $R_0 = 0$ when $F_0(y, x) = 0$;
2) the function $F_0(x, y)$ is sign-definite, and

$$g = F_0(\cos\theta, \sin\theta) \int_0^{2\pi} \frac{R_0(\cos\theta, \sin\theta)}{F_0(\cos\theta, \sin\theta)}\, d\theta = 0$$

The case $R_0 \leqslant 0$ with $F_0(x, y) = 0$ and the case $g < 0$ for the canonical system, can not occur because it leads to asymptotically stable integrals, which contradicts Liouville's theorem.

If, however, the form R_0 with $F_0(x, y) = 0$, takes on positive values, and also if $g > 0$ with a sign-definite $F_0(x, y)$, then the unperturbed motion is unstable.

For the investigation of the cases 1 and 2 it is necessary to consider higher order forms. Let us consider forms of the $m + 1$ order.

Applying the stability criterion, one can prove that for canonical systems in case 1, the stability is possible only when the functions Φ_j vanish if $F_0(x, y) = 0$. In the opposite case we obtain only instability since the case od asymptotic stability can not occur for canonical systems. One reaches the same conclusion considering the case 2.

Investigating forms of order higher than $m+1$, for example forms of the $m+k$ order, we arrive at similar conclusions, i.e. we will obtain either instability or we get the indeterminate case when the problem of stability is not solved by forms of order $m+k$. Thus, just as for rational λ so also for irrational λ, stability can occur only in very special cases.

6. Systems of higher order. Let us consider the system (0.1) and let us assume that all λ_s are irrational and that they do not satisfy any relations of the form

$$\sum_{s=1}^{n} m_s\lambda_s = 0 \qquad \text{for } \sum |m_s| \leqslant N \quad \text{(A)}$$

where the m_s are integers.

Setting $z_s = x_s + iy_s$, $\bar{z}_s = x_s - iy_s$, we obtain

$$z_s^{\cdot} = i\lambda_s z_s + Z_s(z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p, \tau), \qquad \bar{z}_s^{\cdot} = -i\lambda_s \bar{z}_s + \bar{Z}_s(z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p, \tau) \quad (6.1)$$

Here,

$$Z_s = \sum_{l=2}^{\infty} Z_s^{(l)}, \qquad \bar{Z}_s = \sum_{l=2}^{\infty} \bar{Z}_s^{(l)} \qquad (s = 1, \ldots, p)$$

$$Z_s^{**} = \sum A_s^{**}(\tau) z_1^{k_1} \ldots z_p^{k_p} \bar{z}_1^{n_1} \ldots \bar{z}_p^{n_p}$$

$$\bar{Z}_s^{**} = \sum \bar{A}_s^{**}(\tau) \bar{z}_1^{k_1} \ldots \bar{z}_p^{k_p} z_1^{n_1} \ldots z_p^{n_p}$$

$$A_s^{**}(\tau) = A_s^{**}(\tau + 2\pi) \qquad (k_1 + \cdots + k_p + n_1 + \cdots + n_p \geqslant 2)$$

Here, and in the sequel, an asterisk indicates a superscript $(k_1, \ldots, k_p)$ while two asterisks indicate the superscripts $(k, \ldots, k_p, n_1, \ldots, n_p)$.

Passing to the variables ζ_s and $\bar{\zeta}_s$

$$\begin{aligned} \zeta_s &= z_s + \sum u_s^{**}(\tau) z_1^{k_1} \ldots z_p^{k_p} \bar{z}_1^{n_1} \ldots \bar{z}_p^{n_p} \\ \bar{\zeta}_s &= \bar{z}_s + \sum \bar{u}_s^{**}(\tau) \bar{z}_1^{k_1} \ldots \bar{z}_p^{k_p} z_1^{n_1} \ldots z_p^{n_p} \end{aligned} \qquad (6.2)$$

we determine the functions $u_s^{**}(\tau)$ and $\bar{u}_s^{**}(\tau)$ so that on the right-hand side of the transformed system τ does not appear explicitly in the $2N+1$ first forms.

Such a determination of the functions $u_s^{**}(\tau)$ and $\bar{u}_s^{**}(\tau)$ is always possible, whereby these functions are continuous and periodic of period 2π. As a result we obtain

$$\zeta_s^{\cdot} = i\lambda_s\zeta_s + \zeta_s \sum_{k_1+\cdots+k_p \leqslant N} C_s^*(\zeta_1, \bar{\zeta}_1)^{k_1} \cdots (\zeta_p, \bar{\zeta}_p)^{k_p} + P_s(\zeta_1, \ldots, \zeta_p, \bar{\zeta}_1, \ldots, \bar{\zeta}_p, \tau)$$

In this system the C_s^* are constants and the P_s holomorphic functions of ζ_s and $\bar{\zeta}_s$ whose expansions do not contain terms of order less than $2N+2$. The coefficients of these expansions will be continuous and periodic functions in τ of period 2π. The number N may be taken sufficiently large.

Setting

$$\zeta_s = \xi_s + i\eta_s, \qquad C_s^* = \alpha_s^* + i\beta_s^*$$

we obtain

$$\xi_s^{\cdot} = -\lambda_s\eta_s + \xi_s \sum_{k_1+\cdots+k_p \geqslant 1}^{N} \alpha_s^*(\xi_1^2 + \eta_1^2)^{k_1} \cdots (\xi_p^2 + \eta_p^2)^{k_p} -$$

$$- \eta_s \sum_{k_1+\cdots+k_p \geqslant 1}^{N} \beta_s^*(\xi_1^2 + \eta_1^2)^{k_1} \cdots (\xi_p^2 + \eta_p^2)^{k_p} + K_s(\xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_p, \tau) \quad (6.3)$$

$$\eta_s^{\cdot} = \lambda_s\xi_s + \xi_s \sum_{k_1+\cdots+k_p\geqslant 1}^{N} \beta_s^* (\xi_1^2+\eta_1^2)^{k_1}\cdots(\xi_p^2+\eta_p^2)^{k_p} +$$
$$+ \eta_s \sum_{k_1+\cdots+k_p\geqslant 1}^{N} \alpha_s^* (\xi_1^2+\eta_1^2)^{k_1}\cdots(\xi_p^2+\eta_p^2)^{k_p} + L_s(\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_p,\tau)$$

where the K_s and L_s do not contain terms of order less than $2N+2$. Setting $\xi_s = r_s\cos\theta_s$, $\eta_s = r_s\sin\theta_s$, we obtain

$$r_s^{\cdot} = r_s \sum \alpha_s^* r_1^{2k_1}\cdots r_p^{2k_p} + R_s(r_1,\ldots,r_p,\theta_1,\ldots,\theta_p,\tau)$$
$$r_s\theta_s^{\cdot} = \lambda_s r_s + r_s \sum \beta_s^* r_1^{2k_1}\cdots r_p^{2k_p} + F_s(r_1,\ldots,r_p,\theta_1,\ldots,\theta_p,\tau) \qquad (6.4)$$
$$(2\leqslant k_1+\cdots+k_p\leqslant N,\ s=1,\ldots,p)$$

In those cases when the problem of stability is solved by a finite number of terms on the right-hand parts of Equations (6.4), the investigation of the stability of its integrals reduces to the problem on the stability of systems of the pth order with p zero roots, which is represented by the first group of equations of this system. This case was treated in [8].

Let us now consider the case of rational $\lambda_s = \alpha_{s1}/\beta_{s1}$ (α_{s1}, β_{s1} are positive integers). Let the number β be the lowest multiple of all the β_{s1}. Setting $\tau = \beta t$, and passing to the variables ξ_s and η_s by means of Formulas

$$x_s = \xi_s\cos\alpha_s t + \eta_s\sin\alpha_s t, \qquad y_s = \xi_s\sin\alpha_s t - \eta_s\cos\alpha_s t \qquad (\alpha_s = \beta\alpha_{s1}/\beta_{s1})$$

we obtain (6.5)

$$\xi_s^{\cdot} = \sum_{l=2}^{\infty} P_s^{(l)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_p,t), \qquad \eta_s^{\cdot} = \sum_{l=2}^{\infty} Q_s^{(l)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_p,t) \qquad (s=1,\ldots,p)$$

Performing the transformation we get (6.6)

$$x_{s1} = \xi_s + \sum u_s^{**}\xi_1^{k_1}\cdots\xi_p^{k_p}\eta_1^{n_1}\cdots\eta_p^{n_p}$$
$$y_{s1} = \eta_s + \sum v_s^{**}(t)\xi_1^{k_1}\cdots\xi_p^{k_p}\eta_1^{n_1}\cdots\eta_p^{n_p} \quad (2\leqslant k_1+\cdots+k_p+n_1+\cdots+n_p\leqslant N)$$

Let us determine the functions u_s and v_s so that in the transformed system the N first forms should have constant coefficients.

Such a determination of the functions u_s and v_s is always possible. As a result we obtain

$$x_{s1}^{\cdot} = X_{s1}^{(m)}(x_{11},\ldots,x_{p1};y_{11},\ldots,y_{p1}) + \cdots + X_{s1}^{(m+N)}(x_{11},\ldots,x_{p1},y_{11},\ldots,y_{p1}) +$$
$$+ X_{s1}^{(m+N+1)}(x_{11},\ldots,x_{p1};y_{11},\ldots,y_{p1},t) + \cdots$$
$$y_{s1}^{\cdot} = Y_{s1}^{(m)}(x_{11},\ldots,x_{p1};y_{11},\ldots,y_{p1}) + \cdots + Y_{s1}^{(m+N)}(x_{11},\ldots,x_{p1},y_{11},\ldots,y_{p1}) +$$
$$+ Y_{s1}^{(m+N+1)}(x_{11},\ldots,x_{p1};y_{11},\ldots,y_{p1},t) + \cdots$$
$$(m\geqslant 2,\ s=1,\ldots,p) \qquad (6.7)$$

Thus the investigation of the system (6.1) for rational λ_s can be reduced to systems of $2p$ equations with $2p$ zero roots; whereby the forms $X_{s1}^{(l)}$ and $Y_{s1}^{(l)}$ for $l\leqslant m+N$ can have any constant coefficients.

N o t e 1 . If the system (6.1) has μ pairs of pure imaginary roots with irrational λ_s and ν pairs with rational ones ($\mu+\nu=p$) then, combining the transformations (6.2) and (6.6), it is always possible to transform this system to a system of order $\mu+2\nu$ with $\mu+2\nu$ zero roots if the irrational λ_s satisfy the condition (A).

N o t e 2 . Liapunov's problem, and the problem of stability in the case of p pairs of pure imaginary roots are considered here only for the case of critical variables. It is easy to prove that the results remain valid also for the more general case when the system (6.1) has also n roots with negative real parts in addition to p pairs of pure imaginary roots.

We note that the problem of stability in case of irrational λ_s permits considerable simplification in the general case, and that it can be reduced, when $p = 1$, to the investigation of one equation which is obtained from (6.3) by the change of variables $\xi_1 = r\cos\theta$, $\eta_1 = r\sin\theta$. This equation has the form $r^{\cdot} = \alpha^{(m)} r^m + \ldots$ The number $\alpha^{(m)}$ corresponds to a number g which appears in Liapunov's method [1]. When the λ_s are rational, the problem becomes considerably more complicated. The complications do not disappear (as can be seen in Sections 2 to 4) even if $p = 1$ since the investigation of the system (1.6) involves many difficulties.

While it was possible to formulate necessary and sufficient conditions for stability (in terms of forms of the second order) for second order systems (1.6) and while we could analyze more general cases for stability with respect to forms of higher order, it is not possible to find such general condition for stability or instability for systems of the type (6.7).

One criterion of instability with respect to mth order forms for the system (6.7) was obtained by the author in the paper [8]. We shall give it here.

If the system of equations

$$x_s^{\cdot} = X_s^{(m)}(x_1, \ldots, x_n) + X_s^{(m+1)}(x_1, \ldots, x_n) + \cdots \qquad (s = 1, 2, \ldots, n) \qquad (6.8)$$

is such that the forms

$$F_{sk} = x_k X_s^{(m)} - x_s X_k^{(m)}$$

with any fixed k and for $s = 1, 2 \ldots k-1, k+1, \ldots,$ have real solutions different from $x_1 = x_2 = \ldots = x_n = 0$, and if the form

$$R = \sum_{s=1}^{n} x_s X_s^{(m)}(x_1, \ldots, x_n)$$

with $F_{sk} = 0$, can take on positive values, then the unperturbed motion is unstable.

Therefore, in the presence of real roots of the system of equations $F_{sk} = 0$, stability can occur only when for all values of $x_1, \ldots, x_n$, satisfying the condition $F_{sk} = 0$, the expression $R \leqslant 0$.

In case the λ_s are irrational, the system (6.1) can be reduced to the system (6.4) in which the forms $X_s^{(m)}$ are such that the equations $F_{sk} = 0$ take on the form

$$F_{sk} = r_s r_k (R_s^{(m-1)} - R_k^{(m-1)}) = 0$$

$$R_s^{(m-1)} = \sum \alpha^* r_1^{2k_1} \cdots r_p^{2k_p} \qquad (2k_1 + 2k_2 + \cdots + 2k_p = m - 1)$$

System of these equations always has real solutions different from $r_1 = r_2 = \ldots = r_p = 0$.

In case the λ_s are rational, there will always exist such solutions when m is even. If m is odd, the system of equations $F_{sk} = 0$ may have no real solutions besides $x_1 = x_2 = \ldots = x_n = 0$.

From the given criterion for instability with respect to mth order forms it follows that the forms F_{sk} and R play a very important role in problems of stability of the integrals of the system (6 8).

Therefore, it is of interest to obtain a new form of the equations of the perturbed motion whose right-hand sides would contain the forms F_{sk} and R directly. For this purpose let us transform the system (6.8) by setting $x_s = r y_s$ $(s = 1, 2, \ldots, n)$. Suppose that $y_1^2 + \ldots + y_n^2 = 1$. Then $x_1^2 + \ldots + x_n^2 = r^2$.

Differentiating the last equation with respect to t and determining the derivatives of $y_1, \ldots, y_n$ with respect to t by means of (6.8), we obtain a new system of the form

$$\frac{dr}{dt} = r^m R_0 + r^{m+1} R_1 + \cdots$$

$$\frac{dy_s}{dt} = r^{m-1}(y_1 F_{s1}^{(0)} + y_2 F_{s2}^{(0)} + \cdots + y_n F_{sn}^{(0)}) + r^m \sum_{k=1}^{n} y_k F_{sk}^{(1)} + \cdots$$

$$(s = 1, \ldots, n) \tag{6.9}$$

$$\sum_{s=1}^{n} y_s^2 = 1, \quad R_l = \sum_{s=1}^{n} y_s X_s^{(m+l)}(y_1, y_2 \ldots, y_n) \qquad (l = 0, 1, 2, \ldots)$$

$$F_{sk}^{(l)} = y_k X_s^{(m+l)}(y_1, \ldots, y_n) - y_s X_k^{(m+l)}(y_1, \ldots, y_n)$$

We note that

$$F_{ss}^{(l)} \equiv 0, \quad F_{sk}^{(l)} = -F_{ks}^{(l)}$$

Let us now rewrite the first group of Equations (6.4) in the form

$$r_s = r_s R_s^{(m-1)} + r_s R_s^{(m+1)} + \cdots \qquad \left(R_s^{(l)} = \sum_{2k_1 + \cdots + 2k_p = l} \alpha_s^* r_1^{2k_1} \cdots r_p^{2k_p}\right)$$

Setting

$$r_s = r y_s, \quad z_s = y_s^2, \quad \rho = r^2 = \sum_{s=1}^{p} r_s^2, \quad m = 2k + 1$$

and taking into account that

$$F_{sk} = r_s r_k (R_s - R_k) = r_s r_k R_{sk}$$

we obtain

$$\begin{aligned} \frac{d\rho}{dt} &= 2\rho^{k+1} R_0(z_1, \ldots, z_p) + 2\rho^{k+2} R_1(z_1, \ldots, z_p) + \cdots \\ \frac{dz_s}{dt} &= 2\rho^k z_s (z_1 R_{s1}^{(0)} + z_2 R_{s2}^{(0)} + \cdots + z_p R_{sp}^{(0)}) + \cdots \end{aligned} \qquad \begin{pmatrix} s = 1, \ldots, p \\ z_1 + \cdots + z_p = 1 \end{pmatrix} \tag{6.10}$$

Liapunov's function for the system (6.10) may be taken in the form $V = \rho e^{-Nu}$, where u is a continuous bounded function of $z_1, \ldots, z_p$. The derivative of this function with respect to t will have the form

$$V' = 2\rho^{k+1} e^{-Nu} \left\{ \left[R_0 - N \sum \left(\frac{\partial u}{\partial z_s} - \frac{\partial u}{\partial z_k} \right) z_s z_k R_{sk}^{(0)} \right] + \right.$$
$$\left. + \rho \left[R_1 - N \sum \left(\frac{\partial u}{\partial z_s} - \frac{\partial u}{\partial z_k} \right) z_s z_k R_{sk}^{(1)} \right] + \cdots \right\}$$

because of Equations (6.10).

If it should happen that the function u can be found from Equations

$$\frac{\partial u}{\partial z_s} - \frac{\partial u}{\partial z_k} = R_{sk}^{(0)} P_{sk}$$

where P_{sk} are positive continuous bounded functions different from zero or constants, then

$$V' = 2\rho^{k+1} e^{-Nu} [R_0 - N \sum P_{sk} (R_{sk}^{(0)})^2 z_s z_k] + \cdots$$

Let us assume that such a function u has been found. Then the necessary conditions for asymptotic stability with respect to the mth order forms ($R_0 < 0$ with $F_{sk}^{(0)} = 0$) are also sufficient. Let us confine ourselves to the simplest case $p = 2$, i.e. when (6.1) is a system of the fourth order.

In this case the function u is determined by Equation

$$\frac{\partial u}{\partial z_1} - \frac{\partial u}{\partial z_2} = R_{12}^{(0)}, \quad R_{12}^{(0)} = \sum_k a^{(k_1, k_2)} z_1^{k_1} z_2^{k_2} \qquad (k = k_1 + k_2)$$

The function u can be taken in the form

$$u = \sum_{k+1} A^{(k_1,k_2)} z_1^{k_1} z_2^{k_2} \qquad (k+1 = k_1 + k_2)$$

Making the substitution, we determine all the coefficients $A^{(k_1, k_2)}$ in terms of the coefficients $a^{(k_1, k_2)}$.

Returning to the derivative V' whivh, for $p = 2$, has the form

$$V' = 2\rho^{k+1} e^{-Nu} [R_0 - N (R_{12}^{(0)})^2 z_1 z_2] + \cdots$$

we come to the conclusion that if the form $R_0 < 0$ for $F_{12} = z_1 z_2 R_{12}^{(0)} = 0$, then the unperturbed motion is asymptotically stable. This result was obtained in [8]. Considering forms of higher order and selecting the function

$$V = \rho e^{-Nu_1} + \rho^2 e^{u_2} + \cdots + \rho^\alpha e^{u_\alpha}$$

one can obtain criteria of stability in forms of order higher than the mth. We shall, however, not concern ourselves with these questions. We note that the new form of the equations of the unperturbed motion of the kind (6.9), which makes it easier to construct Liapunov's functions for the system (6.8), does not eliminate all difficulties related to their constructions if ~ In these cases one can encounter quite serious difficulties.

7. Canonical systems. Suppose that the system of equations (6.7) is obtained from the system (0.1) under the assumption that its Hamiltonian function has the form

$$H = \sum_{s=1}^{p} \frac{\alpha_s}{2\beta_s} (x_s^2 + y_s^2) + \frac{1}{\beta} \sum_{l=3}^{\infty} H^{(l)} (x_1, \ldots, x_p; y_1, \ldots, y_p, \tau) \qquad (7.1)$$

Here,

$$H^{(l)} = \sum_{n=0}^{\infty} a^{**} (\tau) x_1^{k_1} \cdots x_p^{k_p} y_1^{n_1} \cdots y_p^{n_p}$$

$$a^{**}(\tau) = \sum_{n=0}^{\infty} (\delta_n^{**} \cos n\tau + \gamma_n^{**} \sin n\tau)$$

$$(k_1 + k_2 + \cdots + k_p + n_1 + n_2 + \cdots + n_p = l)$$

In spite of the special nature of the right-hand sides of the obtained system, the investigation of this system is very difficult. The fact is that the canonical systems belong to those special systems for which the stability problem can not be solved by a finite number of forms of the right-hand sides of the systems (6.4) and (6.7). Even though these systems contain N first forms with coefficients independent of time, this circumstance does not simplify the study of the problem because these forms may determine either instability or nonasymptotic stability. If however, one sets $N = \infty$, then one can obtain a self-contained system for forms of any order, and the solution of the problem presents no such difficulties if the transformation series (6.2) and (6.6) are convergent. But these series will diverge in general, and the investigation of their convergence properties presents great difficulties even when $p = 1$.

Avoiding the difficulties connected with the investigation of the stability of these systems, let us consider those canonical systems in which the instability of the unperturbed motion can be revealed by a consideration of the N first forms of the right-hand sides of the system of the perturbed motion. Hereby it is necessary to restrict the investigation to rational λ_s since for irrational λ_s the unperturbed motion will be stable no matter how large the finite number N needs to be chosen.

Let us assume that as a result of the transformation the powest forms $X_s^{(l)}$ and $Y_s^{(l)}$, which appear in the system (6.7) have the index $l = 2$.

We consider the system of algebraic equations (7.2)

$$X_s^{(2)} (\lambda_1, \ldots, \lambda_{k-1}, 1, \lambda_{k+1}, \ldots, \lambda_p, \mu_1, \mu_2, \ldots, \mu_p) = 0 \qquad (\lambda_s = x_s / x_k, \ \mu_s = y_s / x_k)$$

$$Y_s^{(2)} (\lambda_1, \ldots, \lambda_{k-1}, 1, \lambda_{k+1}, \ldots, \lambda_p, \mu_1, \mu_2, \ldots, \mu_p) = 0 \qquad (s = 1, \ldots, p)$$

and the system of equations

$$X_s^{(2)}(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_{k-1}, 1, \mu_{k+1}, \ldots, \mu_p) = 0 \quad (\lambda_s = x_s / y_k, \ \mu_s = y_s / y_k) \quad (7.3)$$

$$Y_s^{(2)}(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_{k-1}, 1, \mu_{k+1}, \ldots, \mu_p) = 0 \quad (s = 1, \ldots, p)$$

For each value of k we shall have two systems of equations of order $2p$ with $2p - 1$ unknowns. If for any fixed index k at least one of these systems does not have real roots, then the unperturbed motion, determined by Hamilton's function (7.1), is unstable.

If these systems have common roots for arbitrary values of k, but the equations $F_{s,k} = 0$, constructed for the system (6.7), have roots different from the common roots of the system (7.2) and (7.3), then the unperturbed motion is also unstable.

If it happens that the forms $X_s^{(l)}$ and $Y_s^{(l)}$ for $l = 2, 3, \ldots, m-1$ vanish identically, and the forms $X_s^{(m)}$ and $Y_s^{(m)}$ are different from zero, then, applying the criteria of instability to this kind of systems, we obtain analogous results for even m.

If m is odd, one has to consider a system of $2p - 1$ algebraic equations $F_{s,k} = 0$ in $2p - 1$ unknowns. If it happens that for one k the system of equations has real solutions, and that for $F_{s,k} = 0$ the form

$$R_0 = \Sigma x_s X_s^{(m)} + \Sigma y_s Y_s^{(m)}$$

can take on positive values, then the unperturbed motion is unstable.

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